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CALCULATION PROCEDURE FOR TRANSIENT
HEAT TRANSFER TO A COOLED PLATE
IN A HEATED STREAM WHOSE TEMPERATURE
VARIES ARBITRARILY WITH TIME

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16. Abstract <p>Solutions for the surface temperature and surface heat flux are found for laminar, constant property, slug flow over a plate, convectively cooled from below, when the temperature of the fluid over the plate varies arbitrarily with time at the plate leading edge. A simple technique is presented for handling arbitrary fluid temperature variation with time by approximating it by a sequence of ramps or steps for which exact analytical solutions are available.</p>			
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CALCULATION PROCEDURE FOR TRANSIENT HEAT TRANSFER TO A COOLED PLATE IN A HEATED STREAM WHOSE TEMPERATURE VARIES ARBITRARILY WITH TIME

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SUMMARY

Solutions for the surface temperature and surface heat flux are found for a flat plate, convectively cooled from below, when the temperature of a fluid flowing over it in a laminar, constant property, slug-flow fashion is varying arbitrarily with time. A specific solution is given for the important case, from the practical standpoint, of the fluid temperature at the plate leading edge varying in a linear fashion with time between two specified temperatures. The solution to the linear case, after being properly generalized, also serves as the basis for the rapid, approximate treatment of transients induced by any fluid temperature variation with time. This is done by approximating the actual fluid temperature variation at the plate leading edge by a series of ramps or steps, thus giving relatively simple functions for the surface heat flux and temperature responses. Several example cases are worked to illustrate the application of the known solutions for the step and the ramp to more general fluid-inlet temperature variations. Finally, a limited comparison with some experimental data is made.

INTRODUCTION

This work predicts the unsteady surface heat flux and temperature of a plate when the fluid passing over it has its temperature at the plate leading edge varying arbitrarily with time. As such, it represents an extension and generalization of the work reported previously in reference 1.

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Transients in a plate are often induced by a controlled change with time of the temperature of the fluid flowing over the plate as opposed to the case of a controlled change in the plate temperature or heat flux. When the fluid induces the transient, which the plate must then respond to, the temperature distributions in the fluid and the plate are mutually coupled, and one has a so-called "conjugate" problem. This plate could be an idealization of a rocket motor wall during startup, a nuclear reactor component, a recuperative heat exchanger during startup and shutdown, or a regenerative heat exchanger either in the transient unsteady domain or, the ultimate periodic unsteady time domain. The primary physical motivation for the work, however, is the transient induced in gas turbine blades and vanes as a result of startup, shutdown, or changes in the steady-state power level of an already operating engine.

A review of much of the previous associated work in transient, forced-convection heat transfer is given in reference 1. The most pertinent of the works reviewed are references 2 and 3, in which heat transfer at a stagnation point due to a sudden change in free-stream temperature, with the wall temperature field mutually coupled to the fluid temperature field, is solved using approximate integral techniques. In reference 4 the dynamic response of thin heat exchanger walls is treated by the Laplace transformation under the assumption of quasi-steady conditions with a constant surface coefficient of heat transfer. Reference 5 treats the case of a transient in a pipe wall (infinite thickness) due to a change in the fluid inlet temperature but restricts the analysis to quasi-steady conditions and to the use of a constant surface coefficient of heat transfer between the wall and the fluid. One of the problems dealt with in reference 6 is the unsteady heat transfer to and temperature of a sphere when it is suddenly subjected to a fluid with a different temperature flowing in a Stokesian manner about the sphere. The entire sphere is lumped in the space coordinates, and its temperature is coupled to that of the fluid. A Laplace transformation of the governing equations, followed by numerical inversion, yields the heat-transfer response curves for the sphere. Quasi-steady solutions are also presented, and their range of validity discussed. In reference 7 a fluid whose temperature is varying sinusoidally with time enters a duct in a slug flow fashion, that is, with a uniform velocity distribution. The duct walls are thin plates, insulated on their outside surfaces and interacting with the fluid on their inside surfaces. After neglecting axial conduction in plate and fluid, lumping the plate temperature transversely, and using the energy balance equation on the plate as a fluid boundary condition, a technique somewhat like the method of complex temperature (ref. 8) is used to effect a solution by separation of variables. Reference 9 also deals with the case of a sinusoidal fluid temperature at the entrance of a duct, but with nonparticipating walls.

The unsteady surface heat flux and temperature of cooled gas turbine blades and vanes, caused by a turbine inlet temperature varying in some prescribed, though completely general, fashion with time, is the subject of the present investigation. Using a

number of idealizations, a solution to the problem is found which allows study of the general heat transfer behavior for laminar flow over blades and vanes (as well as any of the applications previously mentioned) during this type of transient.

ANALYSIS

Arbitrary Variation of Fluid Temperature at $x = 0$ with Time

Consider the plate of thickness b (fig. 1) with its bottom surface subject to a coolant at constant temperature T_c and with a constant surface coefficient of heat transfer h_c between the coolant and the plate bottom. (Symbols are defined in appendix A.) The fluid passing over the plate has its temperature at $x = 0$, the plate leading edge, changing with time t in some prescribed fashion (fig. 2). For $t \leq 0$, $T = T_c$. The response of the surface heat flux and temperature to this arbitrary variation of fluid temperature at $x = 0$ is required.

As in reference 1 the flow is considered to be laminar, low-speed, constant property, two-dimensional planar, and of the thin thermal boundary layer type. In addition, a steady slug flow velocity profile is used, plate and fluid axial conduction is neglected, and the plate temperature is to depend on only x and t (i.e., the plate's temperature is lumped in the y direction). In attempting to find the response to an arbitrary variation with time of the fluid temperature at $x = 0$, one first seeks the solution for a step change in the fluid temperature which then, by virtue of the linearity of the governing equations and boundary conditions, can be used with Duhamel's theorem to yield the more general result. The mathematical statement of the problem for a step change in fluid temperature from T_c to T_0 is

$$\frac{\partial T}{\partial t} + u_\infty \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$$

at $t = 0, x > 0, y > 0, T = T_c$ at $x = 0, t > 0, y > 0, T = T_0$ as $y \rightarrow \infty, t > 0, x > 0, T$ remains finite. The exact analytical solution to this equation, subject to the side conditions shown, was found in reference 1 using double Laplace transformations. The solution is now presented in terms of the more convenient variable (for use with Duhamel's theorem) $\sigma = T - T_c$, with $\sigma_0 = T_0 - T_c$:

$$\sigma = \sigma_0 u(\tau - 1) \left(\operatorname{erf}(Y) + e^{2\eta Y + \eta^2} \{ \operatorname{erf}[\epsilon(\tau - 1) + \eta + Y] - \operatorname{erf}(\eta + Y) \} \right) \quad (1)$$

where the unit step function is

$$u(\tau - 1) = \begin{cases} 0 & \text{for } \tau < 1 \\ +1 & \text{for } \tau \geq 1 \end{cases} \quad (2)$$

The appearance of the unit step function is the mathematical manifestation of the physics of the plane wave phenomenon, namely, that at position x the plate cannot respond to what has happened to the fluid at $x = 0$ and time $= t$ until the fluid, that was at $x = 0$ at time $= t$, reaches the position x . The nondimensional variables appearing in equation (1) are defined in reference 1 and repeated here for convenience.

$$\tau = \frac{u_{\infty} t}{x} \quad Y = \frac{y}{2} \sqrt{\frac{u_{\infty}}{\alpha_f x}} \quad \eta = \frac{h_c}{k_f} \sqrt{\frac{\alpha_f x}{u_{\infty}}} \quad \epsilon = \frac{\rho_f C_{p,f} \sqrt{\frac{\alpha_f x}{u_{\infty}}}}{2 \rho_w C_{p,w} b} \quad (3)$$

Using Duhamel's theorem (ref. 8) along with equation (1) when $\sigma_0 = 1$ and $\beta = \sigma$ gives the transient temperature distribution within the moving fluid for arbitrary variation of the fluid temperature at the plate leading edge with time as

$$\sigma = \int_0^{\tau} \beta(\eta, \epsilon, Y, \tau - \lambda) \frac{d\sigma_{x=0}(\lambda)}{d\lambda} d\lambda \quad (4)$$

where λ is a dummy variable for the nondimensional time τ .

Since the plate surface temperature is usually of primary interest, Y is set equal to zero and the function β is inserted to yield the result for the plate surface temperature when the fluid temperature variation is arbitrary, namely,

$$\sigma_w = e^{\eta^2} \int_0^{\varphi} \{ \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] - \operatorname{erf} (\eta) \} \frac{d\sigma_{x=0}(\lambda)}{d\lambda} d\lambda \quad (5)$$

where $\varphi = \tau - 1$. The treatment of the unit step function during the integration is shown in appendix B. Defining a surface heat flux parameter F as

$$F = \frac{q_w}{k_f \sqrt{\frac{u_\infty}{\pi \alpha_f x}}}$$

and noting that

$$F = - \frac{\sqrt{\pi}}{2} \left(\frac{\partial \sigma}{\partial Y} \right)_{Y=0}$$

allows one to use Duhamel's theorem to get the analogue of equation (5) for the surface heat flux. Thus, the surface heat flux when the fluid temperature at $x = 0$ is varying arbitrarily with time is given as

$$F = - \int_0^\varphi \left\{ e^{-[\epsilon^2(\varphi-\lambda)^2 + 2\eta\epsilon(\varphi-\lambda)]} + \sqrt{\pi} \eta e^{\eta^2} (\text{erf} [\epsilon(\varphi - \lambda) + \eta] - \text{erf} (\eta)) \right\} \frac{d\sigma_{x=0}(\lambda)}{d\lambda} d\lambda \quad (6)$$

When using either general expression (eq. (5) or (6)) one must keep straight the fact that λ is the dummy variable for nondimensional time τ , not for φ . Both integrals, of course, are to be interpreted in the Stieltjes sense.

As a quick check of equation (5) and illustration of its use, it will be applied to the case of a step change in temperature of the fluid at $t = 0$ from T_c to T_0 and then compared with the previously derived result (eq. (15) of ref. 1). At an abrupt change in $\sigma_{x=0}$ at λ_0 , the integral of equation (5) becomes equal to the bracketed portion of the integrand evaluated at λ_0 times the change in $\sigma_{x=0}$ that occurs at λ_0 . For the step change at $t = 0$ one has $\lambda_0 = 0$ and $\Delta\sigma_{x=0} = \sigma_{x=0}^+ - \sigma_{x=0}^-$, where the plus refers to the value just after $t = 0$ and the minus signs refers to the value just before $t = 0$, hence $\Delta\sigma_{x=0} = \sigma_0$. Equation (5) thus becomes

$$\frac{\sigma_w}{\sigma_0} = e^{\eta^2} \{ \text{erf}(\epsilon\varphi + \eta) - \text{erf}(\eta) \} \quad \text{for } \varphi > 0 \quad (7)$$

Rearrangement shows that this equation is identical to equation (15) of reference 1 as it should be.

Solutions for Generalized Step and Ramp in Fluid Temperature at $x = 0$

In general, for an arbitrary $T_{x=0}$ as a function of time, the integrals of equations (5) and (6) often cannot be found analytically. But $T_{x=0}$ can be approximated by a sequence of steps and ramps to any desired accuracy, and equations (5) and (6) do admit analytical, relatively simple solution functions for both the step change and the ramp change in fluid temperature at $x = 0$. Hence, response functions for any arbitrary fluid temperature at $x = 0$ can be constructed, approximately, by the proper combination of the response functions for the generalized step and generalized ramp as will be explained later.

For the response to a generalized step in fluid temperature where $T_{x=0}$ changes from $T_{x=0}^-$ to $T_{x=0}^+$ at time t_i , which corresponds to $\tau_i = \lambda$, equations (5) and (6) yield the solutions (from ref. 10)

$$\Delta\sigma_{w_i}^s = \Delta T_i e^{\eta^2} \{ \operatorname{erf} [\epsilon(\varphi - \tau_i) + \eta] - \operatorname{erf} (\eta) \} \quad (8)$$

for $\varphi > \tau_i$ and

$$F_i^s = -\Delta T_i \left\{ e^{-[\epsilon^2(\varphi - \tau_i)^2 + 2\eta\epsilon(\varphi - \tau_i)]} + \sqrt{\pi} \eta e^{\eta^2} (\operatorname{erf} [\epsilon(\varphi - \tau_i) + \eta] - \operatorname{erf} (\eta)) \right\} \quad (9)$$

for $\varphi > \tau_i$, where $\Delta T_i = T_{x=0}^+ - T_{x=0}^-$ the subscript i on the $\Delta\sigma_w$ and F refers to the time at which the step occurs, and the superscript s indicates that these are responses to a step as opposed to a ramp.

The generalized ramp, in which $T_{x=0}$ changes in a linear fashion from T_i at time t_i to T_j at time t_j , is depicted in the portion of figure 2 bounded by the dashed reference lines. The response functions for this case are important not only for their role when the fluid temperature varies arbitrarily, but also because often during a startup, shutdown, acceleration, or deceleration of a gas turbine engine, the turbine-inlet temperature, as a first approximation, can be represented as varying linearly with time. The equation of the generalized ramp is

$$T_{x=0} = T_i + (T_j - T_i) \left(\frac{t - t_i}{t_j - t_i} \right) \quad t_i \leq t \leq t_j \quad (10)$$

Using this in equations (5) and (6) leads to the following response functions for the ramp (as presented in ref. 10).

$$\Delta\sigma_{w_i}^{r'} = \left(\frac{T_j - T_i}{\tau_j - \tau_i}\right) \frac{e^{\eta^2}}{\epsilon} \left\{ \epsilon(\varphi - \tau_i) \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_i) + \eta] - i^1 \operatorname{erfc}(\eta) \right\} \quad (11)$$

for $\tau_i < \varphi < \tau_j$

$$\Delta\sigma_{w_i}^r = \left(\frac{T_j - T_i}{\tau_j - \tau_i}\right) \frac{e^{\eta^2}}{\epsilon} \left\{ \epsilon(\tau_j - \tau_i) \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_i) + \eta] - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_j) + \eta] \right\} \quad (12)$$

for $\varphi \geq \tau_j$

$$F_i^{r'} = \left(\frac{T_j - T_i}{\tau_j - \tau_i}\right) \frac{\sqrt{\pi} e^{\eta^2}}{\epsilon} \left(\frac{1}{2} \left\{ \operatorname{erf}(\eta) - \operatorname{erf}[\epsilon(\varphi - \tau_i) + \eta] \right\} - \eta \epsilon(\varphi - \tau_i) \operatorname{erfc}(\eta) + \eta \left\{ i^1 \operatorname{erfc}(\eta) - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_i) + \eta] \right\} \right) \quad (13)$$

for $\tau_i < \varphi < \tau_j$ and

$$F_i^r = \left(\frac{T_j - T_i}{\tau_j - \tau_i}\right) \frac{\sqrt{\pi} e^{\eta^2}}{\epsilon} \left(\frac{1}{2} \left\{ \operatorname{erf}[\epsilon(\varphi - \tau_j) + \eta] - \operatorname{erf}[\epsilon(\varphi - \tau_i) + \eta] \right\} - \eta \epsilon(\tau_j - \tau_i) \operatorname{erfc}(\eta) + \eta \left\{ i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_j) + \eta] - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_i) + \eta] \right\} \right) \quad (14)$$

for $\varphi \geq \tau_j$. The subscript i denotes the time at which the ramp begins, the superscript r is the response due to the ramp, the prime on the r indicates the validity of the expression in the φ domain given by $\tau_i < \varphi < \tau_j$, and $i^1 \operatorname{erfc}(\eta)$ is the first repeated integral of the error function and is defined and tabulated in reference 11. (A representative portion of the details of the integrations of eqs. (5) and (6) subject to eq. (10) is given in appendix C.)

Solutions Obtained by Combining Steps and Ramps

Equations (8), (9), and (11) to (14), when appropriately combined, give the transient surface temperature and heat flux due to a prescribed fluid temperature variation at

$x = 0$, when it is approximated by a series of steps and/or ramps. As an illustration, consider the fluid temperature variation at $x = 0$ shown as a solid line in figure 3. Also shown in figure 3 are three dashed lines that form a step and two ramps which approximate fairly closely the actual fluid temperatures as a function of time. (No dashed line is shown beyond time t_3 because it coincides with the solid line there.) Hence, the flux at time t , say, when $t_1 < t < t_2$ is given by the following sum:

$$F = (\text{Eq. (9) with } \tau_i = 0 \text{ and } \Delta T_i = T_1 - T_c) + (\text{Eq. (14) with } \tau_i = 0, \tau_j = \tau_1, \text{ and } T_j \\ - T_i = T_2 - T_1) + (\text{Eq. (13) with } \tau_i = \tau_1, \tau_j = \tau_2, \text{ and } T_j - T_i = T_0 - T_2)$$

The details of the procedure for the case depicted in figure 3 are illustrated in appendix D.

RESULTS AND DISCUSSION

Transient Due To Change From One Steady State Operating Level to Another

In the analysis section and appendix D, the procedure for handling the more general cases of fluid temperature variation with time are discussed, and the example shown in figure 3 is worked in some detail. Now consider the case depicted in figure 4, where $T_{x=0} = T_1$ for $t < 0$ and then linearly increases to T_0 at time $t = t_1$ and is held at T_0 thereafter.

The response of the surface temperature and flux consists of the proper combining of equations (8), (9), and (11) to (14). The contribution to σ_w and F for $t < 0$ can be considered to have resulted from a step change in fluid temperature from T_c to T_1 a long time ago. Hence, this portion of the response is given by equations (8) and (9) as φ (interpreted here as the time elapsed since the step function changed $T_{x=0}$ from T_c to T_1) goes to infinity. Thus, the contributions to the wall temperature and flux for $t > 0$, but due to the history of the fluid before $t = 0$, are

$$\Delta\sigma_w = (T_1 - T_c)e^{\eta^2} \operatorname{erfc}(\eta) \quad (15)$$

and

$$F = -(T_1 - T_c) \sqrt{\pi} \eta e^{\eta^2} \operatorname{erfc}(\eta) \quad (16)$$

For $0 < t < t_1$ the right hand side of equation (11) would be added to equation (15) and the right hand side of equation (13) would be added to equation (16) with τ_i set to zero

and $\tau_j = \tau_1$ with $T_j - T_i = T_0 - T_1$. These sums give the responses in the time domain between 0 and t_1 . For $t > t_1$ the right hand sides of equations (12) and (14) would be added to equations (15) and (16), respectively, to yield surface temperature and heat flux for values of time $t > t_1$. The solution is now complete. Division of the final equations for the σ_w and F by $T_0 - T_c$ to yield $(T_w - T_c)/(T_0 - T_c)$ and Q_w , respectively, gives, as the solution functions for the surface temperature and surface heat flux, for the case shown in figure 4, the following:

$$\frac{T_w - T_c}{T_0 - T_c} = \left(\frac{T_1 - T_c}{T_0 - T_c} \right) e^{\eta^2} \operatorname{erfc}(\eta) + \left(\frac{T_0 - T_1}{T_0 - T_c} \right) \frac{e^{\eta^2}}{\epsilon \tau_1} \left[\epsilon \varphi \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}(\epsilon \varphi + \eta) - i^1 \operatorname{erfc}(\eta) \right] \quad (18)$$

for $0 \leq \varphi \leq \tau_1$

$$\frac{T_w - T_c}{T_0 - T_c} = \left(\frac{T_1 - T_c}{T_0 - T_c} \right) e^{\eta^2} \operatorname{erfc}(\eta) + \left(\frac{T_0 - T_1}{T_0 - T_c} \right) \frac{e^{\eta^2}}{\epsilon \tau_1} \left\{ \epsilon \tau_1 \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}(\epsilon \varphi + \eta) - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] \right\} \quad (19)$$

for $\varphi \geq \tau_1$

$$Q_w = \left(\frac{T_1 - T_c}{T_0 - T_c} \right) \sqrt{\pi} \eta e^{\eta^2} \operatorname{erfc}(\eta) + \left(\frac{T_0 - T_1}{T_0 - T_c} \right) \frac{\sqrt{\pi} e^{\eta^2}}{\epsilon \tau_1} \left\{ \frac{1}{2} [\operatorname{erf}(\epsilon \varphi + \eta) - \operatorname{erf} \eta] + \eta \epsilon \varphi \operatorname{erfc}(\eta) + \eta [i^1 \operatorname{erfc}(\epsilon \varphi + \eta) - i^1 \operatorname{erfc}(\eta)] \right\} \quad (20)$$

for $0 \leq \varphi \leq \tau_1$, and

$$Q_w = \left(\frac{T_1 - T_c}{T_0 - T_c} \right) \sqrt{\pi} \eta e^{\eta^2} \operatorname{erfc}(\eta) + \left(\frac{T_0 - T_1}{T_0 - T_c} \right) \frac{\sqrt{\pi} e^{\eta^2}}{\epsilon \tau_1} \left(\frac{1}{2} \left\{ \operatorname{erf}(\epsilon \varphi + \eta) - \operatorname{erf}[\epsilon(\varphi - \tau_1) + \eta] \right\} + \eta \epsilon \tau_1 \operatorname{erfc}(\eta) + \eta \left\{ i^1 \operatorname{erfc}[\epsilon \varphi + \eta] - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] \right\} \right) \quad (21)$$

for $\varphi \geq \tau_1$.

Equations (18) to (21) are plotted in figures 5 and 6 for a value of $\epsilon\tau_1 = 2.00$ and for $(T_1 - T_c)/(T_0 - T_c) = (T_0 - T_1)/(T_0 - T_c) = 0.50$. The $\epsilon\tau_1$ value is (as noted in ref. 10) a measure of the steepness of the ramp that takes $T_{x=0}$ from T_1 at $t = 0$ to T_0 at $t = t_1$. The curves in figures 5 and 6 are plotted for two values of η : $\eta = 0$ corresponding to an insulated plate (not cooled) and $\eta = 1$ corresponding to a rather well-cooled plate. The plate, of course, does not respond to the change in the fluid temperature at $\tau = 0$ until $\tau = 1$, hence, the reason for plotting the time parameter as $\epsilon(\tau - 1)$. It is seen from figure 5 that, for both cases, $\eta = 0$ and $\eta = 1$, the wall temperature increases monotonically from the original steady-state temperature to the final steady-state temperature corresponding to the new fluid temperature at $x = 0$, T_0 . In figure 6 the surface heat flux continually increases until the displaced time ($\tau - 1$) at which the fluid temperature at $x = 0$ ceases its variation with time. Beyond this time the surface heat flux decays to its final steady state value.

Comparison With Data

A comparison of the predictions of the theory presented in this work with some experimental data would be desirable. However, only an extremely rough comparison could be made because of incomplete data and, in particular, data from a case that did not satisfy the major restrictions of the idealized model. The measured data was from an air cooled vane (described in ref. 12) during an acceleration in a laminar flow region of the vane surface. Because of a nonslug velocity profile, axial and spanwise conduction, transverse temperature gradients, and x -dependent h_c in the experiment, one would not expect even the initial or final steady-state nondimensional temperatures to be predicted well. Hence, an attempt was made to over-ride some of these effects by comparing $(T_w - T_c, \text{initial})/(T_w, \text{initial} - T_c, \text{initial})$ since this parameter is unity at the start of the acceleration for both the measured data and the theoretical solution. The subscript "initial" pertains to values of T_c and T_w just before the acceleration. During the transient in the turbine-inlet temperature, which was approximated by a single ramp for the analysis, one also encounters time varying T_c , h_c , and u_∞ . These parameters are assumed constant and are, in addition to the aforementioned effects, therefore not taken into proper account in the analysis. Also present in the experiment was extreme property variation and a variable thickness wall. To make a rough comparison, arithmetic averages of T_c , u_∞ , and the fluid properties were used along with the value of h_c at the end of the acceleration. Considering the vast differences between the experimental conditions and the constraints on the analytical model, the agreement between theory and experiment is at least encouraging. In figure 7 the measured wall temperature response curve is plotted against time at $x/L = 0.40$ for

an acceleration in which the turbine-inlet temperature changes from 922 to 1644 K (1200° to 2500° F). Also plotted is the analytical curve, which comes from equations (18) and (19). The qualitative behavior of the nondimensional temperature excess ratio just referred to was predicted by the theory. The percentage error between the two was about 26 percent at low times and about 12 percent at the final steady state. The manifold differences between the theoretical model and the experiment preclude any conclusive explanation of these percent differences. However, a rough calculation indicates that the neglect of transverse temperature differences (the lumped in the y direction assumption) is an important difference between the theory and experiment.

SUMMARY OF RESULTS

Expressions for the surface temperature and surface heat flux are found for a plate, convectively cooled from below, while the temperature of a fluid passing over the top of the plate is changed arbitrarily with time at the plate leading edge.

1. Simple, easy to use functions were presented that allow a solution to the problem of fluid temperature varying arbitrarily with time by approximating the arbitrary fluid temperature variation by a sequence of ramps or steps.
2. The mechanics of properly combining these functions to yield plate temperature and heat flux is demonstrated by means of examples.
3. Equations are given for the important case of a ramp in fluid temperature at $x = 0$ causing $T_{x=0}$ to change from temperature T_1 at time $t = 0$ to temperature T_0 at $t = t_1$ and held at T_0 thereafter.
4. A limited comparison of the predictions of the idealized model with experimental results was made for a case that did not satisfy many of the restrictions of the model. This yielded a rough qualitative similarity of the predictions and the data.

Lewis Research Center,
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APPENDIX A

SYMBOLS

b	plate thickness
C_p	constant-pressure specific heat
F	surface heat flux parameter, $\frac{q_w}{k_f \sqrt{\frac{u_\infty}{\pi \alpha_f x}}}$
F_i^r	surface heat flux response function defined by eq. (14)
$F_i^{r'}$	surface heat flux response function defined by eq. (13)
F^s	surface heat flux response function defined by eq. (9)
h_c	coolant-side surface coefficient of heat transfer
$i^1 \operatorname{erfc}(z)$	first repeated integral of error function, $\int_z^\infty \operatorname{erfc}(z') dz'$
k_f	thermal conductivity of fluid flowing over upper surface of plate
Q_w	instantaneous nondimensional surface heat flux
q_w	instantaneous surface heat flux
T	temperature
T_c	coolant temperature
$T_{x=0}$	instantaneous fluid temperature at $x = 0$
t	time
$u(\tau - 1)$	unit step function, equal to 0 for $\tau < 1$, and to +1 for $\tau \geq 1$
u_∞	free-stream velocity
v	variable defined by eq. (C5)
x	space coordinate along plate
Y	nondimensional y coordinate defined in eqs. (3)
y	space coordinate perpendicular to plate
α_f	thermal diffusivity of fluid flowing over plate
β	value of σ when $\sigma_0 = 1$
ϵ	nondimensional variable defined in eqs. (3)
ρ	density

λ	dummy variable for time
σ	temperature difference, $T - T_c$
$\Delta\sigma_{w_i}^r$	temperature response function defined by eq. (12)
$\Delta\sigma_{w_i}^{r'}$	temperature response function defined by eq. (11)
$\Delta\sigma_{w_i}^s$	temperature response function defined by eq. (8)
η	nondimensional variable defined in eqs. (3)
τ	nondimensional time defined in eqs. (3)
τ_1	nondimensional duration of ramp that begins at $t = 0$
φ	shifted nondimensional time, $\tau - 1$

Subscripts:

f	properties of fluid flowing over plate
i	index
j	index
w	wall
0	ultimate steady-state fluid condition at $x = 0$
∞	free-stream conditions

Superscripts:

r	response to ramp after it has ended
r'	response to ramp before it has ended
s	response to step in fluid temperature at $x = 0$

APPENDIX B

DEVELOPMENT OF EQUATION (5)

To find σ_w for an arbitrary variation of fluid temperature at $x = 0$, against time, when the wall is initially at T_c , Y is set equal to 0 in equation (4), and β is given by the right side of equation (1) with $Y = 0$, $\sigma_0 = +1$, and τ replaced by $\tau - \lambda$. Doing this yields

$$\sigma_w = e^{\eta^2} \int_0^\tau u(\tau - \lambda - 1) \left\{ \operatorname{erf} [\epsilon(\tau - \lambda - 1) + \eta] - \operatorname{erf} (\eta) \right\} \frac{d\sigma_{x=0}(\lambda)}{d\lambda} d\lambda \quad (B1)$$

However, $u(\tau - \lambda - 1) = +1$ only for $\tau - \lambda - 1 \geq 0$ by definition of the unit step function; hence, $\lambda \leq \tau - 1$. Thus, since λ is the integration variable and values of $\lambda > \tau - 1$ lead to a zero integral, this is taken into account by making the upper limit equal to $\tau - 1$, which has been defined as φ . This change obviates the need for explicit appearance of the unit step function in equation (B1) and thus it becomes

$$\sigma_w = e^{\eta^2} \int_0^\varphi \left\{ \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] - \operatorname{erf} (\eta) \right\} \frac{d\sigma_{x=0}(\lambda)}{d\lambda} d\lambda \quad (B2)$$

which is equation (5).

APPENDIX C

DEVELOPMENT OF EQUATION (11)

Inserting equation (10) into equation (5) when $\tau_i < \varphi < \tau_j$ gives, after rearrangement,

$$\Delta\sigma_{w_i}^{r'} = \left(\frac{T_j - T_i}{\tau_j - \tau_i} \right) e^{\eta^2} \int_{\tau_i}^{\varphi} \{ \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] - \operatorname{erf} (\eta) \} d\lambda \quad (C1)$$

Define the integral as I:

$$I = \int_{\tau_i}^{\varphi} \{ \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] - \operatorname{erf} (\eta) \} d\lambda \quad (C2)$$

The second term of equation (C2) can be immediately integrated to give the following expression for I:

$$I = \int_{\tau_i}^{\varphi} \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] d\lambda - (\varphi - \tau_i) \operatorname{erf} (\eta) \quad (C3)$$

Define

$$I_1 = \int_{\tau_i}^{\varphi} \operatorname{erf} [\epsilon(\varphi - \lambda) + \eta] d\lambda \quad (C4)$$

To facilitate the integration in equation (C4), the following variable change is made:

$$v = \epsilon(\varphi - \lambda) + \eta \quad (C5)$$

Therefore,

$$d\lambda = - \frac{dv}{\epsilon}$$

and this variable change causes the limits on the integral to be mapped as follows: τ_i corresponds (from eq. (C5)) to $\epsilon(\varphi - \tau_i) + \eta = v$ and φ corresponds to $v = \eta$, so

equation (C4) becomes

$$I_1 = -\frac{1}{\epsilon} \int_{\epsilon(\varphi - \tau_i) + \eta}^{\eta} \operatorname{erf}(v) dv = \frac{1}{\epsilon} \int_{\eta}^{\epsilon(\varphi - \tau_i) + \eta} \operatorname{erf}(v) dv \quad (C6)$$

Adding and subtracting unity to the integral of equation (C6), noting that $1 - \operatorname{erf}(v) = \operatorname{erfc}(v)$, and integrating the constant term gives

$$I_1 = \varphi - \tau_i - \frac{1}{\epsilon} \int_{\eta}^{\epsilon(\varphi - \tau_i) + \eta} \operatorname{erfc}(v) dv \quad (C7)$$

Rearranging gives

$$I_1 = \varphi - \tau_i - \frac{1}{\epsilon} \left[\int_{\eta}^{\infty} \operatorname{erfc}(v) dv - \int_{\epsilon(\varphi - \tau_i) + \eta}^{\infty} \operatorname{erfc}(v) dv \right] \quad (C8)$$

However, the remaining integrals are simply the first repeated integral of the error function (ref. 11), and thus equation (C8) becomes

$$I_1 = \varphi - \tau_i + \frac{1}{\epsilon} \left\{ i^1 \operatorname{erfc} [\epsilon(\varphi - \tau_i) + \eta] - i^1 \operatorname{erfc} (\eta) \right\} \quad (C9)$$

Substituting equation (C9) into (C3) and then equation (C3) into (C1) yields the final result

$$\Delta \sigma_{w_i}^{r'} = \left(\frac{T_j - T_i}{\tau_j - \tau_i} \right) \frac{e^{\eta^2}}{\epsilon} \left\{ \epsilon(\varphi - \tau_i) \operatorname{erfc} (\eta) + i^1 \operatorname{erfc} [\epsilon(\varphi - \tau_i) + \eta] - i^1 \operatorname{erfc} (\eta) \right\} \quad (C10)$$

for $\tau_i < \varphi < \tau_j$. This is equation (11).

APPENDIX D

MECHANICS OF SURFACE TEMPERATURE FUNCTION SOLUTION FOR FLUID TEMPERATURE VARIATION SHOWN IN FIGURE 3

The solid line of figure 3 shows the variation with time of the fluid temperature at $x = 0$. The surface temperature response as a function of time is desired. Rather than using the actual fluid temperature against time in equation (5) and then numerically integrating the result, the approximate procedure stressed in this report will be used. Hence, the actual fluid temperature variation is approximated by a step and two ramps as shown by the dashed lines in figure 3. Now equations (8), (11), and (12) must be properly combined to yield the surface temperature as a function of time.

First consider the time period $0 < t < t_1$, which corresponds to $0 < \tau < \tau_1$ and, because $\varphi = \tau - 1$, also to $0 \leq \varphi \leq \tau_1$. In this time period the fluid temperature at $x = 0$ is still on the first ramp; hence, the plate has only felt the effect of the step change and the portion of the first ramp that has occurred. Hence, its temperature response is the sum of equations (8) and (11), properly interpreted. (Note that eq. (12) is not yet needed because φ is less than τ_j not greater. Also, no response function for the second ramp is needed because the second ramp has not yet occurred.) Since the step occurs at $t = 0$, it follows that $\tau_i = 0$ for the step and that $\Delta T_i = T_1 - T_c$ in equation (8). Also, the first ramp begins at $t = 0$ or $\tau_i = 0$ and ends at t_1 , which gives $\tau_1 = u_\infty t_1 / x$ and $(T_j - T_i) = (T_2 - T_1)$. Utilizing this information in equations (8) and (11) and then adding them gives the wall temperature response for $0 \leq \varphi < \tau_1$ as

$$\sigma_w = (T_1 - T_c) e^{\eta} \left[\operatorname{erf}(\epsilon\varphi + \eta) - \operatorname{erf}(\eta) \right] + \left(\frac{T_2 - T_1}{\epsilon\tau_1} \right) e^{\eta} \left[\epsilon\varphi \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}(\epsilon\varphi + \eta) - i^1 \operatorname{erfc}(\eta) \right] \quad (D1)$$

In the time interval $t_1 \leq t < t_2$, corresponding to $\tau_1 \leq \varphi < \tau_2$ (where $\tau_2 = u_\infty t_2 / x$) the surface temperature response consists of the response to the step, which is simply the first term of equation (D1), the response due to ramp 1, and the response due to the portion of ramp 2 that has occurred. Since ramp 1 is over, $\varphi > \tau_j$, which means $\varphi > \tau_1$ for ramp 1; and equation (12) applies with $\tau_i = 0$, $\tau_j = \tau_1$, and $T_j - T_i = T_2 - T_1$. For the response to ramp 2, equation (11) applies, since for this ramp $\tau_i = \tau_1$, $\tau_j = \tau_2$, and $\tau_1 \leq \varphi < \tau_2$. Also for ramp 2, $T_j - T_i = T_0 - T_2$. With these substitutions, the sum of the first term of equation (D1) (eq. (8) really) and equa-

tions (12) and (11) give the surface temperature in the nondimensional time domain $\tau_1 \leq \varphi < \tau_2$ as follows:

$$\begin{aligned}\sigma_w = & (T_1 - T_c)e^{\eta^2} [\operatorname{erf}(\epsilon\varphi + \eta) - \operatorname{erf}(\eta)] + \left(\frac{T_2 - T_1}{\epsilon\tau_1} \right) e^{\eta^2} \left\{ \epsilon\tau_1 \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}(\epsilon\varphi + \eta) \right. \\ & \left. - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] \right\} + \left(\frac{T_0 - T_2}{\tau_2 - \tau_1} \right) \frac{e^{\eta^2}}{\epsilon} \left\{ \epsilon(\varphi - \tau_1) \operatorname{erfc}(\eta) \right. \\ & \left. + i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] - i^1 \operatorname{erfc}(\eta) \right\} \quad (D2)\end{aligned}$$

For times $t > t_2$ corresponding to $\varphi > \tau_2$, the surface temperature response is due to the step at $\tau = 0$, the first term of equation (D2), the ramp between 0 and τ_1 , which is now complete, the second term of equation (D2), and the ramp between τ_1 and τ_2 , which is also complete. This last contribution is given by equation (12) with $T_j - T_i = T_0 - T_2$, $\tau_i = \tau_1$, and $\tau_j = \tau_2$. Nothing else has to be added beyond τ_2 since there the fluid temperature at $x = 0$ is a constant. Hence, adding the first two terms of equation (D2) to equation (12) gives the surface temperature in the time domain $\varphi > \tau_2$ as

$$\begin{aligned}\sigma_w = & (T_1 - T_c)e^{\eta^2} [\operatorname{erf}(\epsilon\varphi + \eta) - \operatorname{erf}(\eta)] + \left(\frac{T_2 - T_1}{\epsilon\tau_1} \right) e^{\eta^2} \left\{ \epsilon\tau_1 \operatorname{erfc}(\eta) + i^1 \operatorname{erfc}(\epsilon\varphi + \eta) \right. \\ & \left. - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] \right\} + \left(\frac{T_0 - T_2}{\tau_2 - \tau_1} \right) \frac{e^{\eta^2}}{\epsilon} \left\{ \epsilon(\tau_2 - \tau_1) \operatorname{erfc}(\eta) \right. \\ & \left. + i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_1) + \eta] - i^1 \operatorname{erfc}[\epsilon(\varphi - \tau_2) + \eta] \right\} \quad (D3)\end{aligned}$$

The procedure for finding the surface heat flux function F is essentially the same as that which led to equations (D1) to (D3), except that equations (9), (13), and (14) will be used instead of equations (8), (11), and (12). To demonstrate the steady-state form of σ_w for this case, and also as a partial check on equation (D3), let $\varphi \rightarrow \infty$ in equation (D3). Since $\operatorname{erf}(\infty) = +1$, $1 - \operatorname{erf}(\eta) = \operatorname{erfc}(\eta)$, and $i^1 \operatorname{erfc}(\infty) \rightarrow 0$, equation (D3) reduces to, after cancelling and combining terms,

$$\sigma_w \rightarrow (T_0 - T_c) e^{\eta^2} \operatorname{erfc}(\eta) \quad \text{as } \varphi \rightarrow \infty$$

This is also what equation (7) reduces to as $\varphi \rightarrow \infty$, and this is as it should be, since at long times after the last disturbance is complete (ramp 2 in this case as shown in fig. 3), the wall "forgets" the nature of the disturbances which changed $T_{x=0}$ from T_c to T_0 . Hence the steady-state solution should be, and is, independent of the way in which $T_{x=0}$ goes from T_c to T_0 .

It should now be apparent how the procedure used here on a step and two ramps is to be used when any number of steps or ramps are chosen to approximate an arbitrary fluid temperature at $x = 0$ as a function of time t .

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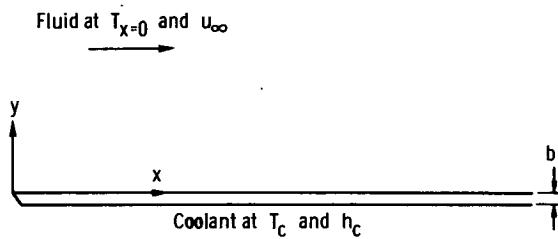


Figure 1. - Sketch of physical situation showing coordinate system.

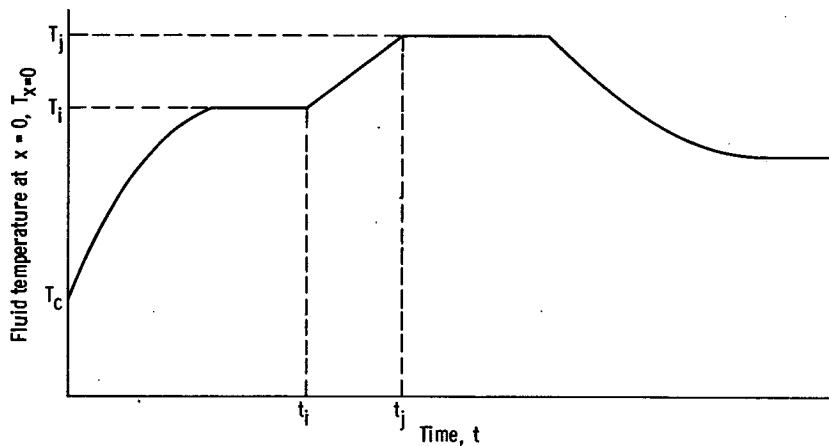


Figure 2. - Arbitrary variation with time of the fluid temperature.

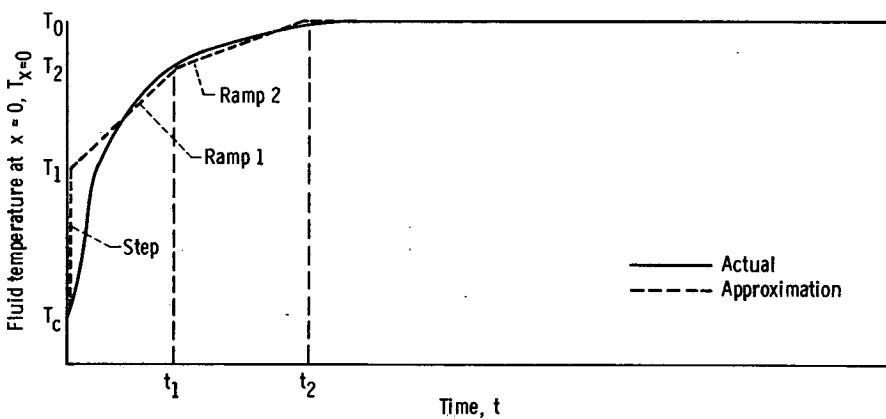


Figure 3. - Actual fluid temperature against time and its approximation by combining a step and two ramps.

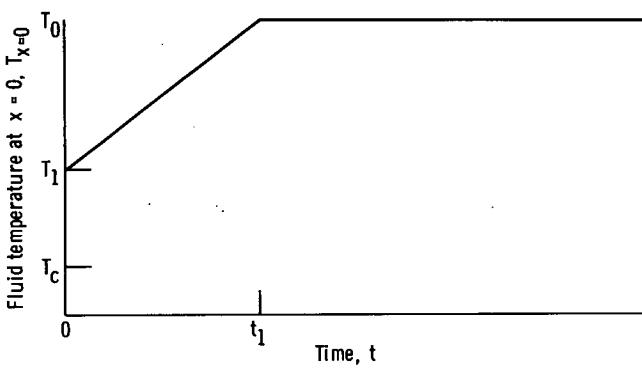


Figure 4. - Ramp change in fluid inlet temperature to an ultimate constant value T_0

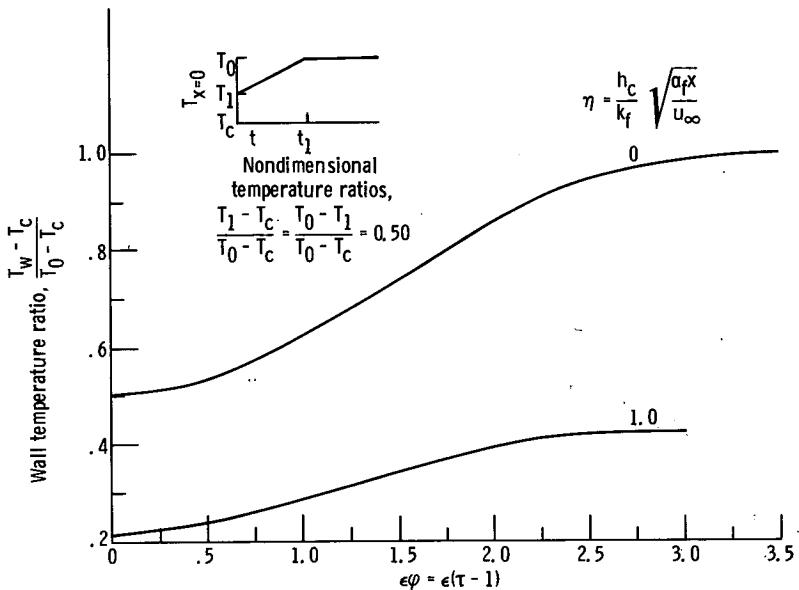


Figure 5. - Wall temperature against shifted time for a ramp change of inlet temperature for the $T_{x=0}$ given in figure 4. Ramp steepness, $\epsilon\tau_1 = 2.00$.

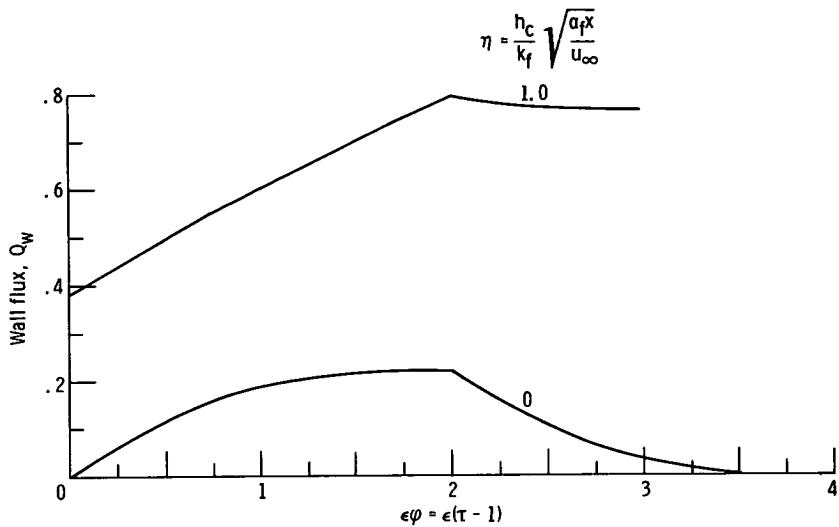


Figure 6. - Wall flux against shifted time for a ramp change of inlet temperature. Ramp steepness, $\epsilon\tau_1 = 2.00$; nondimensional temperature ratios, $(T_1 - T_c)/(T_0 - T_c) = (T_0 - T_1)/(T_0 - T_c) = 0.50$.

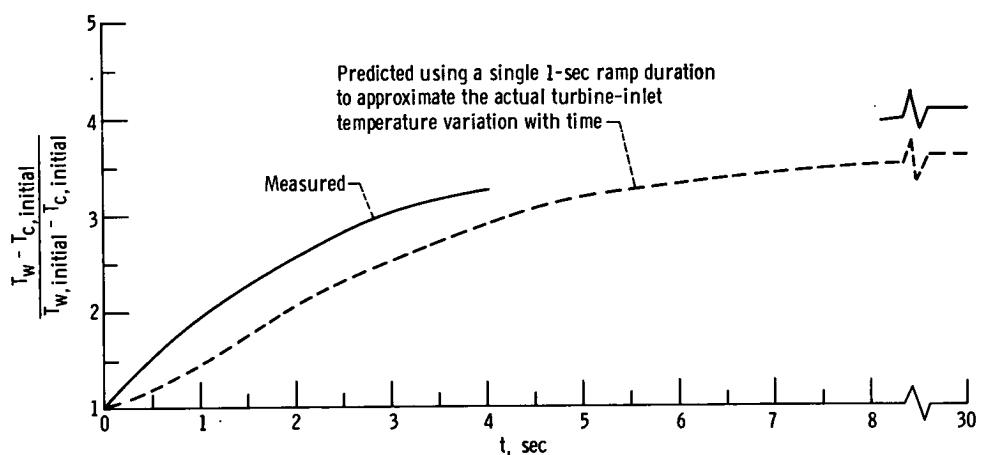


Figure 7. - Comparison of measured and predicted wall temperature variation with time at $x/L = 0.40$.

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